Euclidean wormholes in 2d CFT from quantum chaos and number theory

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2309.00611, 2309.02533: with W. Reeves, M. Rozali

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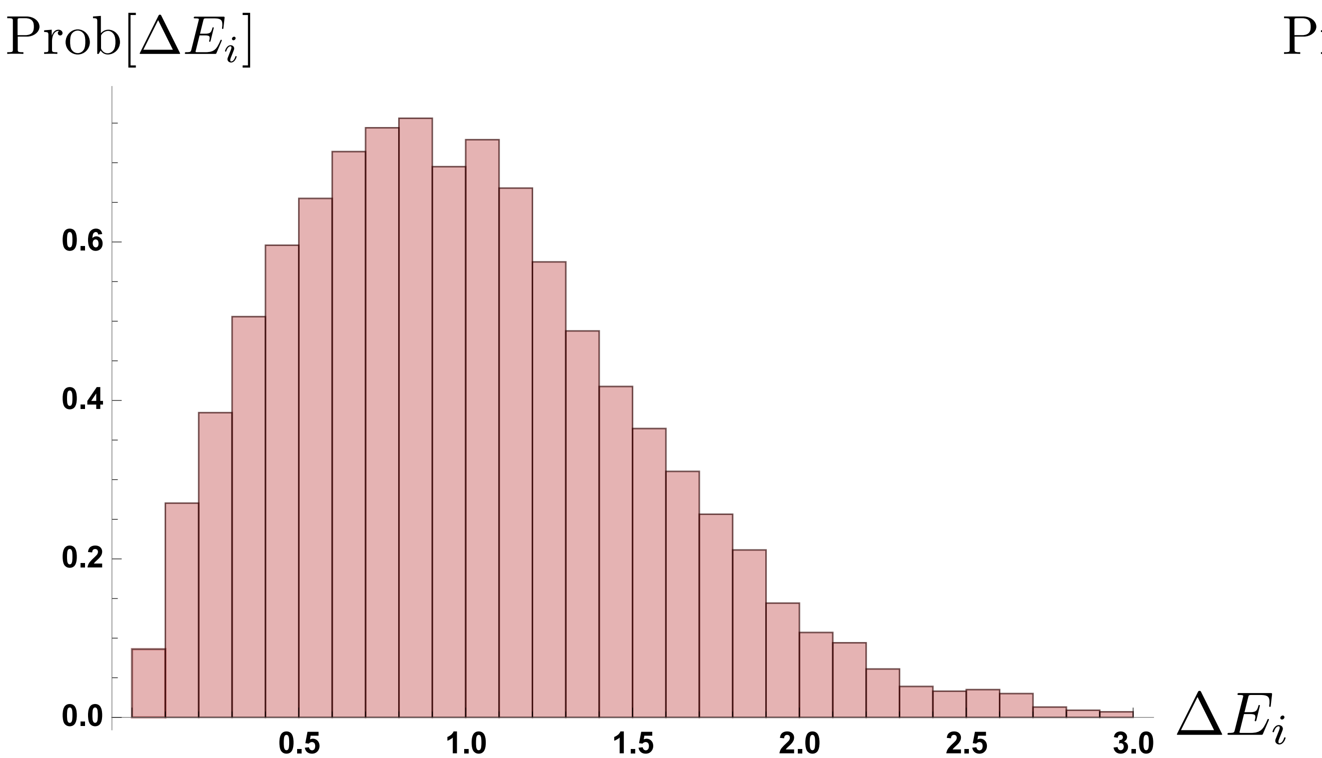
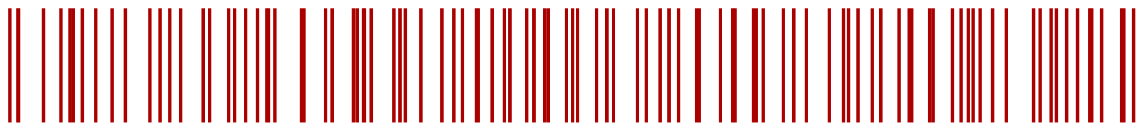
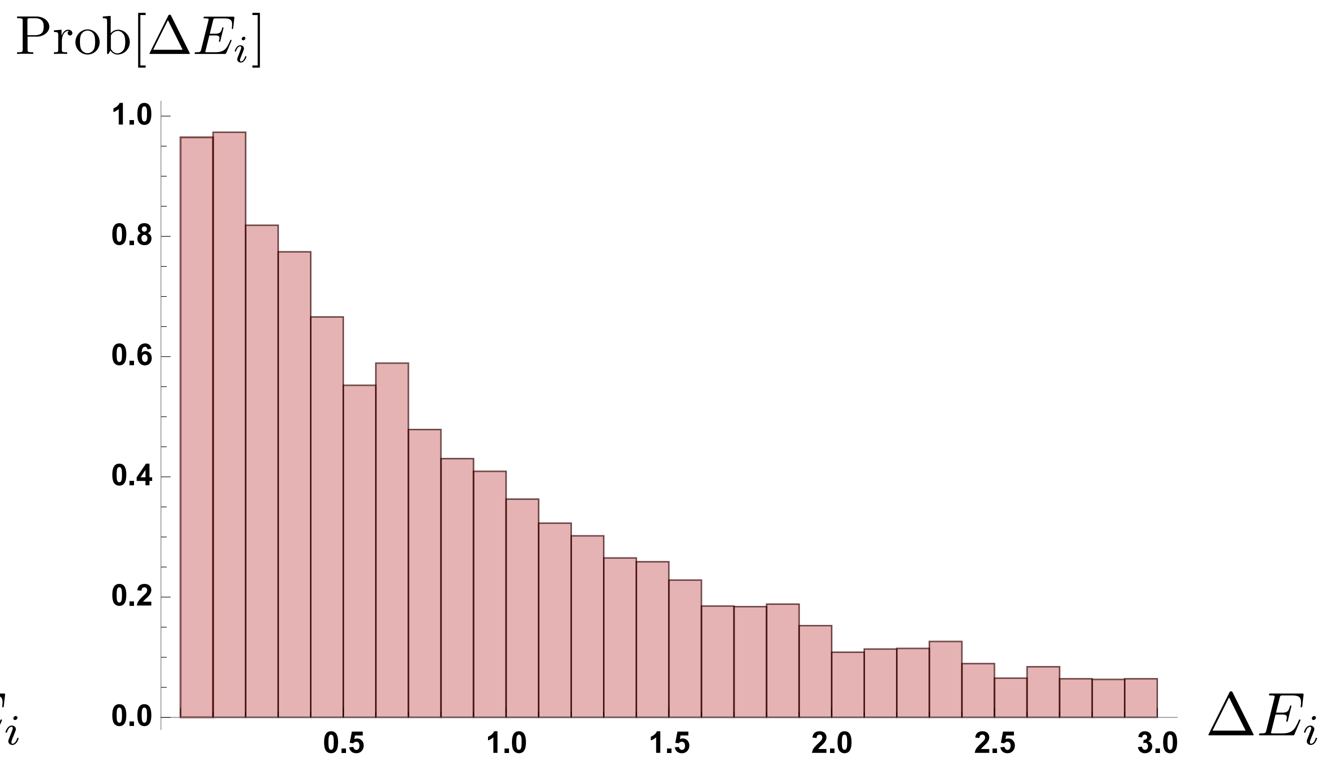
Signatures of chaos

- Universal aspect of quantum chaos: RMT energy level repulsion

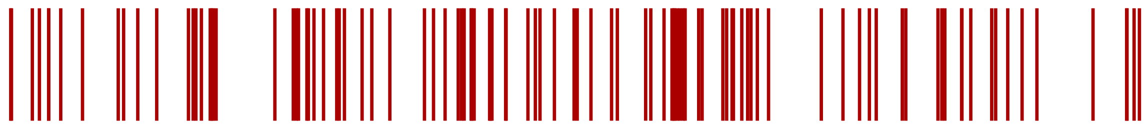


GOE matrix eigenvalues

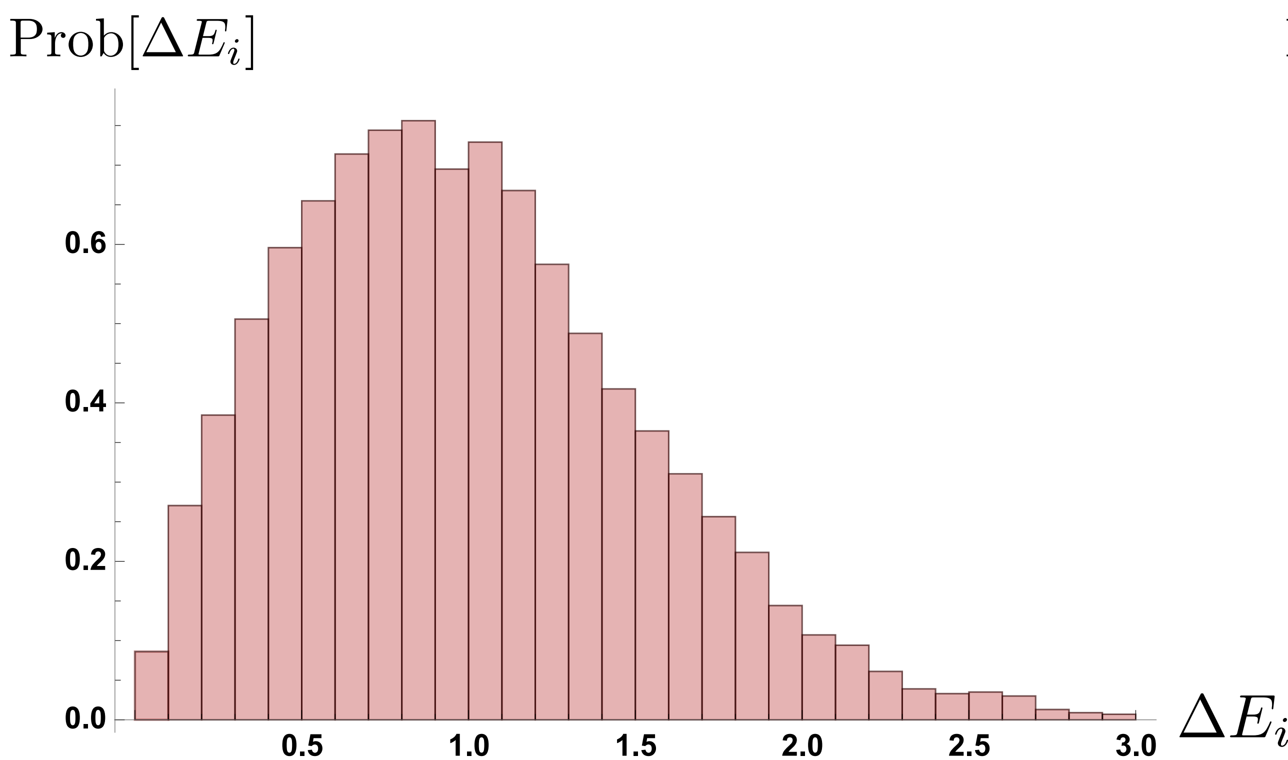




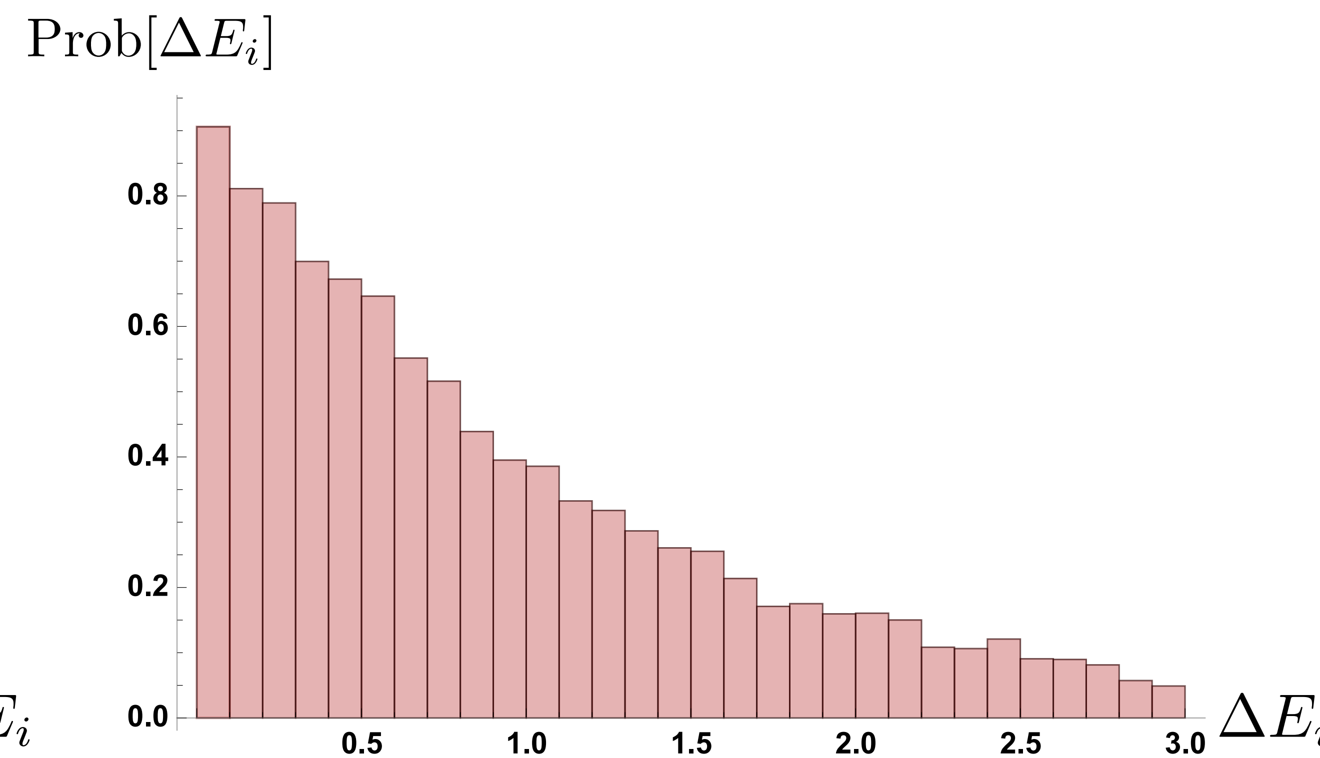
Poisson random numbers



- Symmetries obscure chaotic statistics



single GOE matrix



10 GOE blocks

$$\begin{pmatrix} & [\mathrm{GOE}]_{N\times N} & \end{pmatrix}$$

$$\begin{pmatrix} [\mathrm{GOE}]_{\frac{N}{10}\times\frac{N}{10}} & & & \\ & [\mathrm{GOE}]_{\frac{N}{10}\times\frac{N}{10}} & & \\ & & \ddots & \\ & & & [\mathrm{GOE}]_{\frac{N}{10}\times\frac{N}{10}} \end{pmatrix}$$

- With symmetries: focus on fixed “charge sector” to see chaos

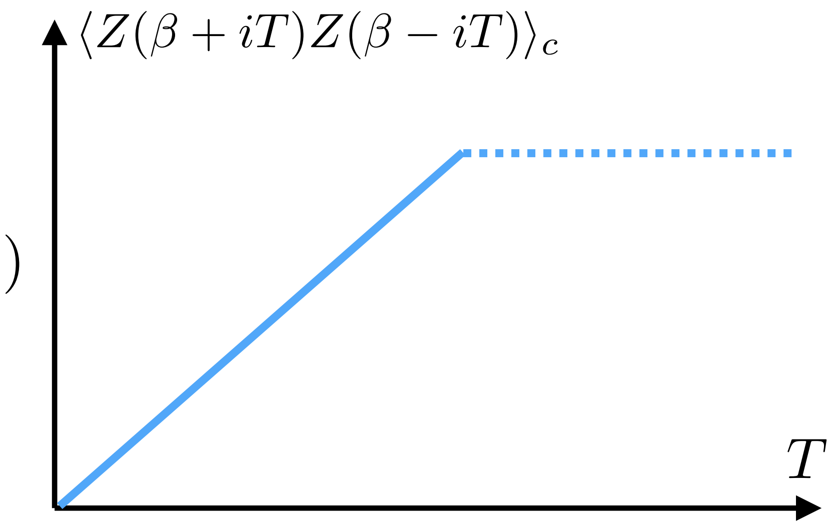
Chaos in 2d CFT

- RMT universality:

$$\langle \rho(E_1)\rho(E_2)\rangle \sim -\frac{1}{\pi^2|\omega|^2} \qquad (\omega \equiv E_1 - E_2 \ll E_i - E_{m_i} \ll 1)$$

$$\langle Z(\beta+iT)Z(\beta-iT)\rangle \sim \frac{T}{2\pi\beta} \qquad (T \gg \beta \gg 1)$$

"linear ramp"

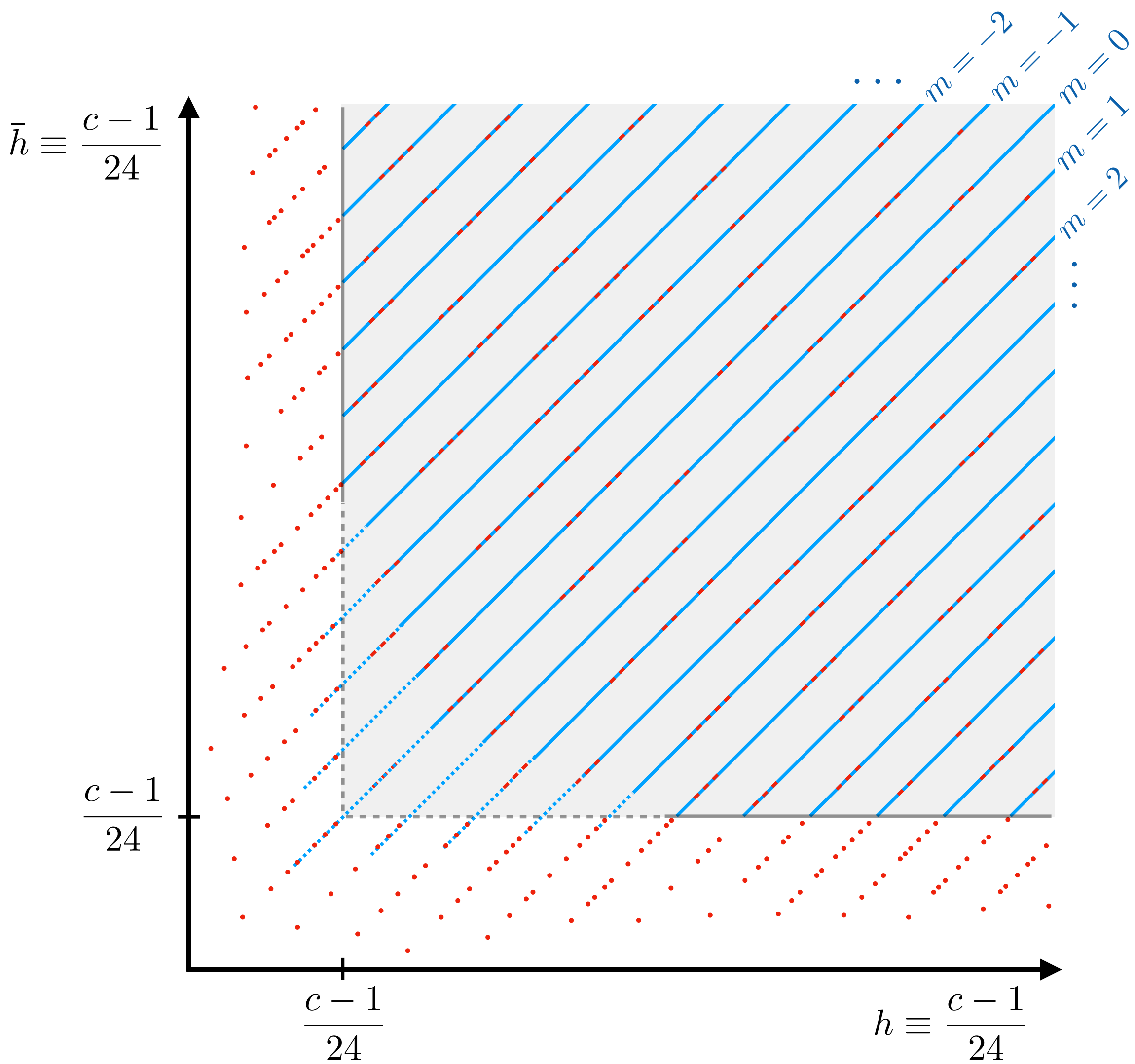


- Important in holography (black holes, SYK, ...) [Cotler + 8][Garcia-Garcia/Verbaarschot] [Saad/Shenker/Stanford][Stanford/Witten]...

- What about 2d CFT? [Cotler/Jensen] [Belin/de Boer] [Chandra/Collier/Hartman/Maloney] ...

  - Problem: huge amount of symmetry/constraints!
  - How to extract RMT universality?

- Consider part of 2d CFT spectrum unconstrained by symmetries



1. **Primary partition function:**

$$Z_{\rm P}(\tau \equiv x+iy) = y^{1/2}|\eta(\tau)|^2\, Z(\tau)$$

2. **Consider states above "black hole threshold":** $\min(h,\bar{h}) > \frac{c-1}{24}$

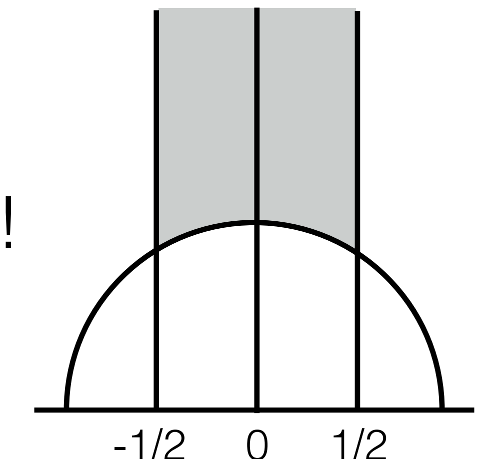
   - Remove states below threshold and their 'modular completion'
     [Benjamin/Collier/Fitzpatrick/Maloney/Perlmutter '21]

$$\widetilde{Z}_{\rm P} = Z_{\rm P} - \hat{Z}_{\rm light}$$

e.g.: $\hat{Z}_{\rm light} = \sum_{\gamma \in SL(2,\mathbb{Z})/\Gamma_\infty} Z_{\rm light}(\gamma\tau)\,, \quad Z_{\rm light} = \sum_{\substack{h,\bar{h}: \\ \min(h,\bar{h}) \leq \frac{c-1}{24}}} q^{h-\frac{c-1}{24}}\, \bar{q}^{\bar{h}-\frac{c-1}{24}}$

3. **Fix spin** $m = h - \bar{h}$ **:** $\widetilde{Z}_{\rm P}^m(y) \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, \widetilde{Z}_{\rm P}(x+iy)\, e^{2\pi i m x}$

- What about modular invariance? $[E, E+\delta E]$ not invariant… :(
- Express spectrum in basis where modular invariance is **manifest**!
  - Spanned by eigenfunctions of Laplacian on $\mathcal{F} = \mathbb{H}/SL(2,\mathbb{Z})$



1. Real-analytic Eisenstein series (“scattering states”): $\Delta_{\mathcal{F}} E_s = s(1-s) E_s$

$$E_{s=\frac{1}{2}+i\alpha} = \sqrt{y}\left(y^{i\alpha} + \frac{\Lambda(i\alpha)}{\Lambda(-i\alpha)}\, y^{-i\alpha} \right) + \sum_{m>0} \cos(2\pi m x)\, \frac{2\, a_m^{(\alpha)}}{\Lambda(-i\alpha)}\, \sqrt{y} K_{i\alpha}(2\pi m y)$$

$s \in \frac{1}{2} + i\mathbb{R}$

$\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$
completed zeta-function

2. Maass cusp forms (“bound states”): $\Delta_{\mathcal{F}} \nu_n = \left(\frac{1}{4} + R_n^2\right) \nu_n$

$$\nu_n = \sum_{m>0} \cos(2\pi m x)\, a_m^{(n)}\, \sqrt{y} K_{iR_n}(2\pi m y)$$

$n \in \mathbb{Z}_+$

Fourier coefficients — eigenvalues

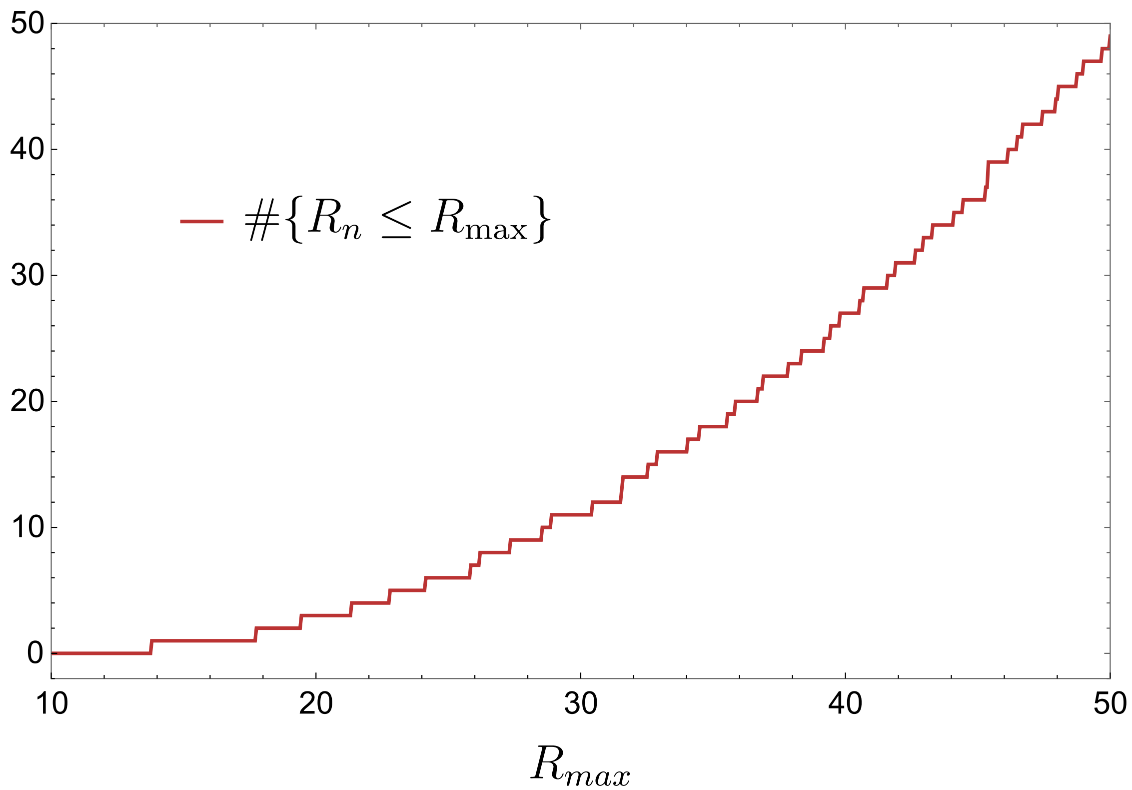
“cusp form data”
(not known analytically;
interesting math conjectures)

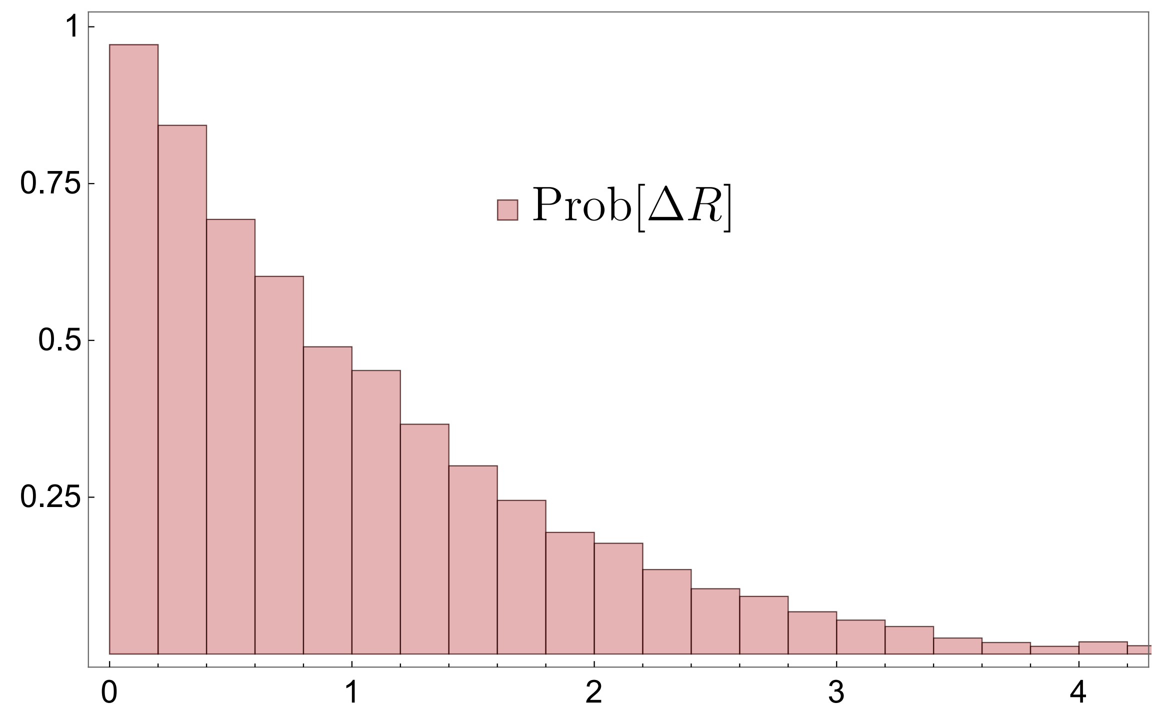
- Cusp forms are "random" linear combination of Bessel functions

$$\nu_n = \sum_{m>0} \cos(2\pi m x)\, a_m^{(n)}\, \sqrt{y} K_{iR_n}(2\pi m y)$$

- Eigenvalues R_n : sporadic numbers, Poisson distributed

$$R_n = 13.7798.., \quad 17.7386.., \quad 19.4235.., \quad 21.3158.., \quad 22.7859.., \quad 24.1124.., \quad 25.8262.., \; \ldots$$





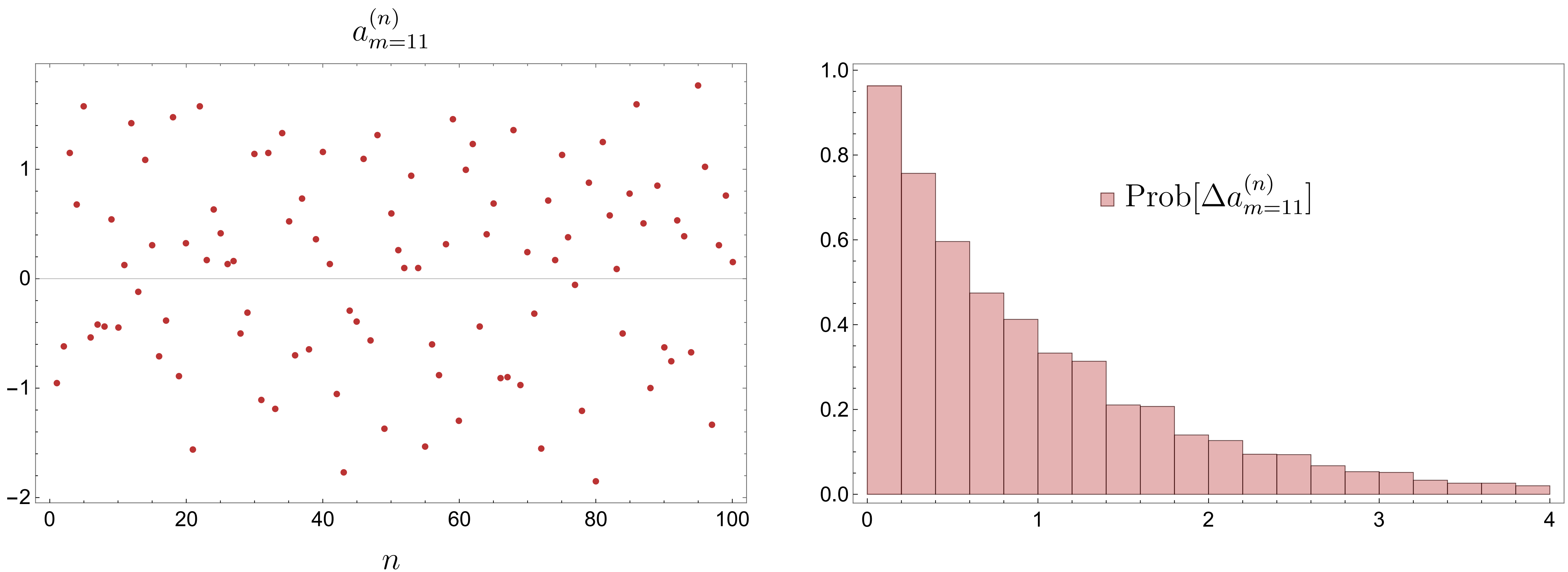
[Then '04] [2309.00611]

- Cusp forms are "random" linear combination of Bessel functions

$$\nu_n = \sum_{m>0} \cos(2\pi m x)\, a_m^{(n)}\, \sqrt{y} K_{iR_n}(2\pi m y)$$

- Eigenvalues R_n : sporadic numbers, Poisson distributed
- Fourier coefficients $a_m^{(n)}$: also erratic, Poisson

$$a_{m=11}^{(n)} = -0.954.., \quad -0.621.., \quad +1.154.., \quad +0.678.., \quad +1.575.., \quad -0.533.., \quad \ldots$$



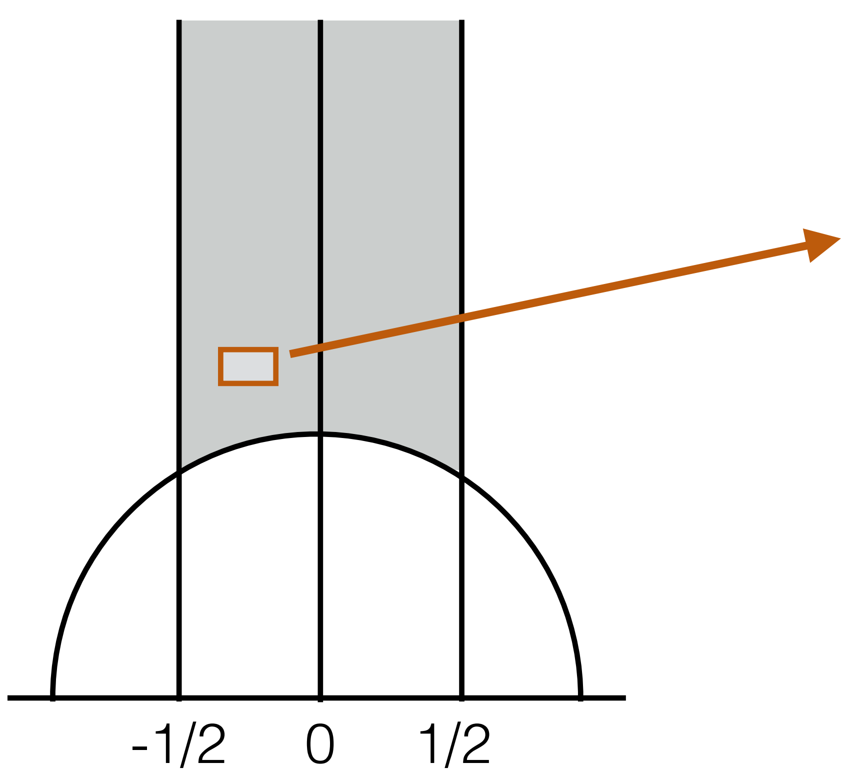
[Then '04] [2309.00611]

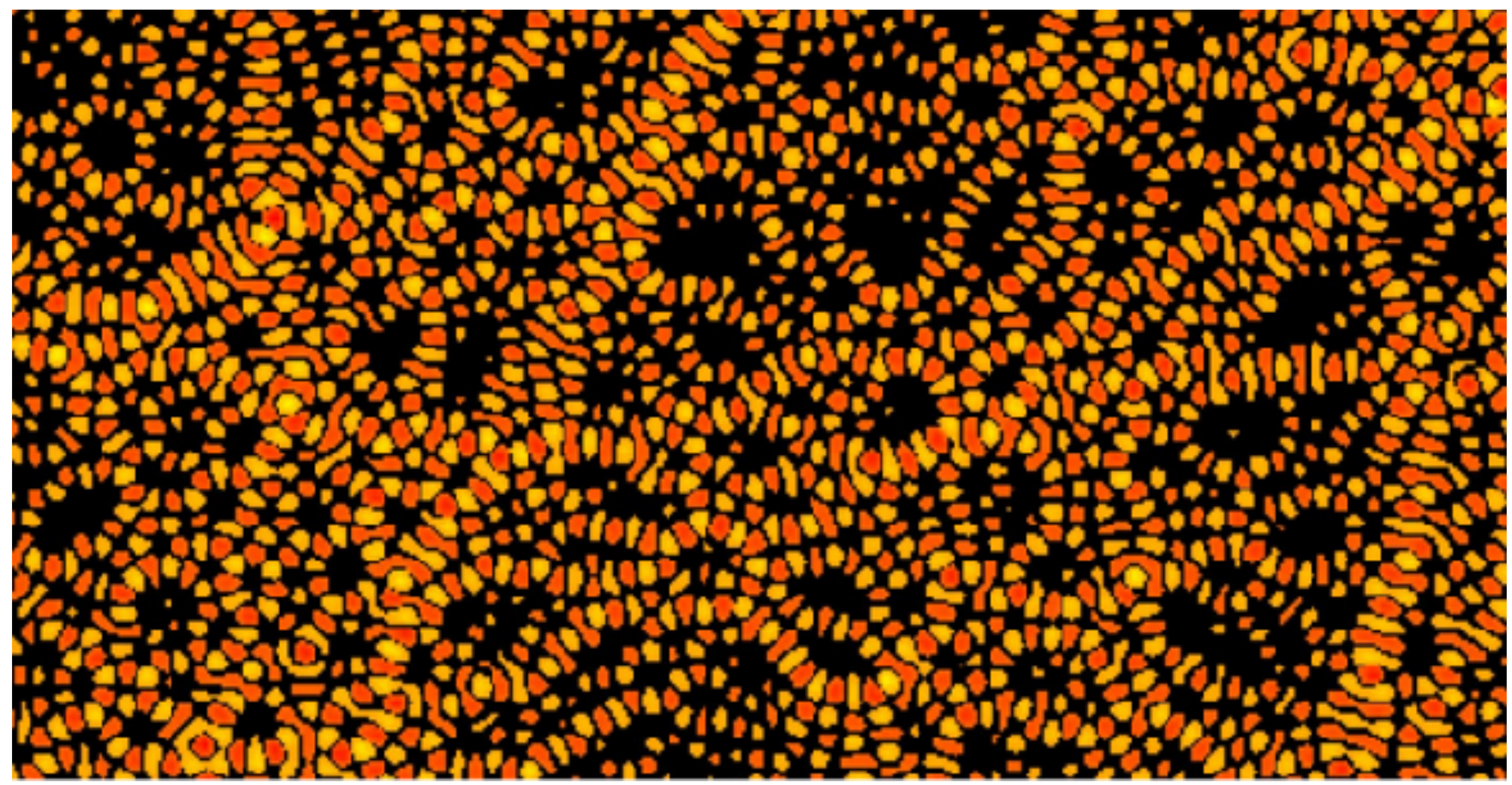
- "Arithmetic chaos" [Sarnak '93] [Hejhal/Arno '93] [Steil '94]
- Random, but subject to infinitely many constraints (Hecke relations)

- Cusp forms are “random” linear combination of Bessel functions

$$\nu_n = \sum_{m>0} \cos(2\pi m x)\, a_m^{(n)}\, \sqrt{y} K_{iR_n}(2\pi m y)$$

e.g.: cusp form with $R_n = 40000.000164..$





[H. Then, Math. Comp. 74 (2004)]

Spectral decomposition

$$\widetilde{Z}_{\rm P} \in L^2(\mathcal{F}) = \mathbb{C} \oplus L^2_{\rm cont.(Eisenstein)} \oplus L^2_{\rm disc.(cuspidal)}$$

- Can expand in modular invariant basis:

[Rankin '39] [Selberg '40] ...
[Benjamin/Collier/Fitzpatrick/Maloney/Perlmutter '21]

$$\widetilde{Z}_{\rm P}(\tau) = \langle \widetilde{Z}_{\rm P} \rangle + \frac{1}{4\pi i} \int_{s\equiv\frac{1}{2}+i\mathbb{R}} ds \, \underbrace{(\widetilde{Z}_{\rm P}, E_s)}_{z_{s\equiv\frac{1}{2}+i\alpha}} E_s(\tau) + \sum_{n\geq 1} \underbrace{(\widetilde{Z}_{\rm P}, \overline{\nu}_n)}_{z_n} \overline{\nu}_n(\tau)$$

$$\overline{\nu}_n = \frac{\nu_n}{\|\nu_n\|}$$

- Spectral form factor: correlations in spectral overlap coefficients

$$\left\langle \widetilde{Z}_{\rm P}(\tau_1)\widetilde{Z}_{\rm P}(\tau_2)\right\rangle_c = \frac{1}{(4\pi i)^2} \iint_{\frac{1}{2}+i\mathbb{R}} ds_1 ds_2 \left\langle z_{s_1} z_{s_2}\right\rangle E_{s_1}(\tau_1) E_{s_2}(\tau_2) + \sum_{n_1,n_2\geq 1} \left\langle z_{n_1} z_{n_2}\right\rangle \overline{\nu}_{n_1}(\tau_1)\,\overline{\nu}_{n_2}(\tau_2)$$

  - $\langle z_{s_1} z_{s_2}\rangle$, $\langle z_{n_1} z_{n_2}\rangle$: unconstrained by modular invariance
  - What does the universal 'ramp' look like in this basis?

Spectral decomposition of the ‘ramp’

- Start with spin 0 ramp:

$$\langle \widetilde{Z}_{\rm P}^{m=0}(y_1)\widetilde{Z}_{\rm P}^{m=0}(y_2)\rangle \sim \frac{1}{\pi}\,\frac{y_1 y_2}{y_1+y_2} + \ldots \qquad \left(y_i \gg 1\,,\quad \frac{y_1}{y_2} = \text{fix}\right)$$

[… analog of $\langle Z(\beta+iT)Z(\beta-iT)\rangle \sim \dfrac{T}{2\pi\beta} \qquad (T \gg \beta \gg 1)$]

[… recall $Z_{\rm P}(\tau_i) \sim y_i^{-1/2}|\eta(\tau_i)|^2 Z(\tau_i)$]

- Start with spin 0 ramp:

$$\langle \widetilde{Z}_{\rm P}^{m=0}(y_1)\widetilde{Z}_{\rm P}^{m=0}(y_2)\rangle \sim \frac{1}{\pi}\frac{y_1 y_2}{y_1+y_2}+\dots \qquad \left(y_i \gg 1\,,\quad \frac{y_1}{y_2}=\text{fix}\right)$$

- Only Eisenstein series have a spin 0 component —> encode spin 0 'ramp'

$$\left\langle z_{s_1\equiv\frac{1}{2}+i\alpha_1}\, z_{s_2\equiv\frac{1}{2}+i\alpha_2}\right\rangle_{\rm ramp} \sim \frac{1}{2\cosh(\pi\alpha_1)}\times 4\pi\,\delta(\alpha_1+\alpha_2)+(\text{subleading}) \qquad (\alpha_i\to\infty)$$

↑ "diagonal"

[2302.14482]
[Di Ubaldo/Perlmutter '23]

- Can check:

$$\frac{1}{(4\pi i)^2}\iint_{\frac{1}{2}+i\mathbb{R}} ds_1 ds_2\,\langle z_{s_1}\, z_{s_2}\rangle_{\rm ramp}\, E_{s_1}(\tau_1)E_{s_2}(\tau_2)\sim \frac{1}{\pi}\frac{y_1y_2}{y_1+y_2}+\sum_{m>0}\cos(2\pi m x)\times(\text{subleading})$$

↑ spin 0 'ramp'

↑ "imprint" of spin 0 ramp onto other spin sectors

- Importantly, the imprint of the spin 0 ramp onto spin *m* is not a ramp! Otherwise chaos in different spin sectors wouldn't be independent.

- Ramp for arbitrary spins m_1, m_2:

[Di Ubaldo/Perlmutter '23]
[2309.00611]
[2309.02533]

subleading corrections

encodes ramp for spin 0

$$\frac{\delta_{m_1 m_2}}{\pi}\frac{y_1 y_2}{y_1+y_2}e^{-2\pi|m_1|(y_1+y_2)}+\mathcal{G}_{m_1 m_2}=\frac{1}{4\pi i}\int_{s=\frac{1}{2}+i\mathbb{R}}ds\left[\frac{1}{2\cosh(\pi\alpha)}+\ \dots\ \right]E_s^{m_1}(y_1)\,E_s^{m_2}(y_2)$$

$$+\sum_{n\geq 1}\left[\frac{1}{2\cosh(\pi R_n)}+\ \dots\ \right]\frac{\nu_n^{m_1}(y_1)}{\|\nu_n\|}\frac{\nu_n^{m_2}(y_2)}{\|\nu_n\|}$$

encodes ramp for spins >0

schematically: $$[\text{ramp}]+[\text{subleading}]=\mathrm{Tr}_{L^2(\mathcal{F})}\begin{pmatrix}\frac{1}{2\cosh(\pi\alpha)} & 0 & \cdots \\ 0 & \frac{1}{2\cosh(\pi\alpha)} & \\ \vdots & & \ddots\end{pmatrix}$$

$$L^2(\mathcal{F})=\mathbb{C}\oplus L^2_{\text{cont.(Eisenstein)}}\oplus L^2_{\text{disc.(cuspidal)}}$$

- This spectral decomposition follows from **Kuznetsov trace formula**

$$\delta_{m_1 m_2} \frac{\sqrt{y_1 y_2}}{\pi^2} \int_{\mathbb{R}} d\alpha\, \alpha \tanh(\pi\alpha)\, g(\alpha)\, K_{i\alpha}(2\pi m_1 y_1) K_{i\alpha}(2\pi m_2 y_2) + \mathcal{G}_{m_1 m_2}[g]$$
$$= \int_{\mathbb{R}} \frac{d\alpha}{4\pi} \frac{g(\alpha)}{2\cosh(\pi\alpha)} E^{m_1}_{\frac{1}{2}+i\alpha}(y_1) E^{m_2}_{\frac{1}{2}+i\alpha}(y_2) + \sum_{n\geq 1} \frac{g(R_n)}{2\cosh(\pi R_n)} \frac{\nu_n^{m_1}(y_1)}{\|\nu_n\|} \frac{\nu_n^{m_2}(y_2)}{\|\nu_n\|}$$

[Bruggeman '78]
[Kuznetsov '81]
[2309.02533]

schematically:

$$\underbrace{\delta_{m_1 m_2} \int d\alpha\, g(\alpha)(\cdots) + \mathcal{G}_{m_1 m_2}[g]}_{\text{geometric decomposition}} = \underbrace{\mathrm{Tr}_{L^2(\mathcal{F})} \begin{pmatrix} \frac{g(\alpha)}{2\cosh(\pi\alpha)} & 0 & \cdots \\ \vdots & \frac{g(\alpha)}{2\cosh(\pi\alpha)} & \\ & & \ddots \end{pmatrix}}_{\text{spectral decomposition}}$$

geometric decomposition
(related to Poincare sum over
SL(2,Z) images of some seed)

spectral decomposition

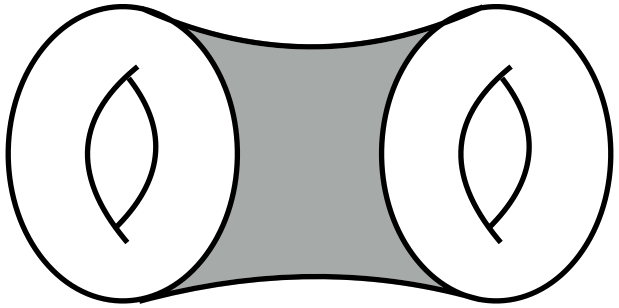
- This spectral decomposition follows from **Kuznetsov trace formula**

$$\delta_{m_1 m_2} \frac{\sqrt{y_1 y_2}}{\pi^2} \int_{\mathbb{R}} d\alpha\, \alpha \tanh(\pi\alpha)\, g(\alpha)\, K_{i\alpha}(2\pi m_1 y_1) K_{i\alpha}(2\pi m_2 y_2) + \mathcal{G}_{m_1 m_2}[g]$$
$$= \int_{\mathbb{R}} \frac{d\alpha}{4\pi} \frac{g(\alpha)}{2\cosh(\pi\alpha)} E^{m_1}_{\frac{1}{2}+i\alpha}(y_1) E^{m_2}_{\frac{1}{2}+i\alpha}(y_2) + \sum_{n\geq 1} \frac{g(R_n)}{2\cosh(\pi R_n)} \frac{\nu_n^{m_1}(y_1)}{\|\nu_n\|} \frac{\nu_n^{m_2}(y_2)}{\|\nu_n\|}$$

[Bruggeman '78]
[Kuznetsov '81]
[2309.02533]

- 'Bare ramp': $g(\alpha) = 1 \quad \Rightarrow \quad \mathrm{LHS} = [\mathrm{ramp}] + \mathcal{G}_{m_1 m_2}$

  - "Remainder" $\mathcal{G}_{m_1 m_2}$: matches corrections found in the **[torus]x[interval] AdS3 wormhole amplitude**
    [Cotler/Jensen '20]



$$\mathcal{G}_{m_1 m_2}[g(\alpha) = 1] \;=\; \mathcal{G}^{(\mathrm{AdS}_3 \text{ wormhole})}_{m_1 m_2}$$

Gravity amplitude = simplest (minimal) completion of the 'bare ramp' into a SFF consistent with modular invariance

"MaxRMT" principle
[Di Ubaldo/Perlmutter '23]

- Ramp is encoded in “random sum of cusp forms”! How??

$$\frac{1}{\pi}\frac{y_1 y_2}{y_1+y_2}\, e^{-2\pi m (y_1+y_2)} + \ldots \; = \; \sum_{n\geq 1} \frac{1}{2\,\cosh(\pi R_n)} \frac{\nu_n^m(y_1)\nu_n^m(y_2)}{\|\nu_n\|^2}$$

‘arithmetic chaos’

‘ramp’
(quantum chaos)

$$\approx 2\,\frac{\sqrt{y_1 y_2}}{\pi^2}\int_0^\infty d\alpha\,\alpha\,\tanh(\pi\alpha) K_{i\alpha}(2\pi m y_1) K_{i\alpha}(2\pi m y_2)$$

↑
?

- Ramp is encoded in “random sum of cusp forms”! How??

$$\frac{1}{\pi}\frac{y_1 y_2}{y_1+y_2}\, e^{-2\pi m(y_1+y_2)} + \ldots \;=\; \sum_{n\geq 1} \frac{1}{2\cosh(\pi R_n)} \frac{\nu_n^m(y_1)\nu_n^m(y_2)}{\|\nu_n\|^2}$$

‘arithmetic chaos’

‘ramp’
(quantum chaos)

$$= \sum_{n\geq 1} \frac{4}{L^{(n)}_{\nu\times\nu}(1)}\, \nu_n^m(y_1)\nu_n^m(y_2)$$

$$L^{(n)}_{\nu\times\nu}(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{m\geq 1} \frac{\left(a_m^{(n)}\right)^2}{m^s}$$

$$= \prod_{p \text{ prime}} \frac{1}{1-\left(a_p^{(n)}\right)^2\left(p^{-s}-p^{-2s}\right)+\left(p^{-s}-p^{-2s}-p^{-3s}\right)}$$

$$= \sum_{n\geq 1} 4 \prod_{p \text{ prime}} \left[1-\left(a_p^{(n)}\right)^2\left(p^{-1}-p^{-2}\right)+\left(p^{-1}-p^{-2}-p^{-3}\right)\right] \times \left(a_m^{(n)}\right)^2 \sqrt{y_1 y_2}\, K_{iR_n}(2\pi m y_1) K_{iR_n}(2\pi m y_2)$$

- Ramp is encoded in "random sum of cusp forms"! How??

$$\frac{1}{\pi}\frac{y_1 y_2}{y_1+y_2}\, e^{-2\pi m(y_1+y_2)} + \ldots = \sum_{n\geq 1} 4 \prod_{p \text{ prime}} \left[1-\left(a_p^{(n)}\right)^2\left(p^{-1}-p^{-2}\right)+\left(p^{-1}-p^{-2}-p^{-3}\right)\right] \times \left(a_m^{(n)}\right)^2 \sqrt{y_1 y_2}\, K_{iR_n}(2\pi m y_1) K_{iR_n}(2\pi m y_2)$$

  - Large $y_{1,2}$: larger R_n dominate, which are also more dense ($\bar{\mu}(R) \propto R$)

$$\Rightarrow \qquad \sum_n f(R_n) \xrightarrow{\approx} \int_{R_0}^{\infty} dR\, \bar{\mu}(R)\, f(R)$$

  - **Statistical average** over sum of cusp forms! In particular:

$$\overline{\prod_{p \text{ prime}} \left[1-\left(a_p^{(n)}\right)^2\left(p^{-1}-p^{-2}\right)+\left(p^{-1}-p^{-2}-p^{-3}\right)\right] \times \left(a_m^{(n)}\right)^2} = \frac{6}{\pi^2}$$

$\uparrow$

    - **This is exact!** Contains **full information about statistical distribution** of Fourier coefficients $a_m^{(n)}$ for all spins [2309.00611]
    - For large *n*: "1" is required in order to get the universal ramp
    - Corrections to "1": theory-dependent subleading corrections

- **Statistical average** over sum of cusp forms! In particular:

$$\overline{\prod_{p\ \mathrm{prime}} \left[1-\left(a_p^{(n)}\right)^2\left(p^{-1}-p^{-2}\right)+\left(p^{-1}-p^{-2}-p^{-3}\right)\right]\times\left(a_m^{(n)}\right)^2} = \frac{6}{\pi^2}$$

<u>Proof:</u>

- Prime decomposition:

$$m = p_1^{k_1}\cdots p_r^{k_r} \qquad \Rightarrow \qquad \left(a_m^{(n)}\right)^2 = \left(a_{p_1^{k_1}}^{(n)}\right)^2\cdots\left(a_{p_r^{k_r}}^{(n)}\right)^2$$

- Hecke algebra:

$$a_m^{(n)}a_{m'}^{(n)} = \sum_{\substack{\ell|(m,m')\\ \ell>0}} a_{\frac{mm'}{\ell^2}}^{(n)} \qquad \text{e.g.:} \qquad \begin{aligned} a_{p_1^{k_1}\cdots p_r^{k_r}}^{(n)} &= a_{p_1^{k_1}}^{(n)}\cdots a_{p_r^{k_r}}^{(n)} \\ a_{p^k}^{(n)} &= a_{p^{k-1}}^{(n)}\,a_p^{(n)}-(1-\delta_{k,1})\,a_{p^{k-2}}^{(n)} \end{aligned}$$

- Distributions of prime spin coefficients:

$$\boldsymbol{\mu}_p(a) = \begin{cases} \frac{(p+1)\sqrt{4-a^2}}{2\pi\left(\left(p^{1/2}+p^{-1/2}\right)^2-a^2\right)} & \text{if } |a|<2 \\ 0 & \text{otherwise} \end{cases}$$

E.g. if m prime: only need $\overline{\left(a_p^{(n)}\right)^2} = \frac{p+1}{p}\,,\quad \overline{\left(a_p^{(n)}\right)^4} = \frac{2p^2+3p+1}{p^3}$

For generic m : need all moments of $\boldsymbol{\mu}_p(a)$

[2309.00611]

Summary

Summary

- Main assumption: 2d CFT exhibits RMT universality (linear ramp).
- Ramp encodes statistical information about arithmetic chaos of Maass cusp forms. For AdS_3 $\mathbb{T}^2 \times I$ wormhole this information is **maximal**.
- Kuznetsov trace formula determines subleading corrections, required by modular invariance
  - Spectral correlations proportional to 'identity matrix' on $L^2(\mathcal{F}) \times L^2(\mathcal{F})$
  - **Minimal** corrections: $\mathbb{T}^2 \times I$ wormhole

Questions

- What properties of CFT spectrum imply a ramp?
- EFT (Efetov sigma-model) of the ramp?
- Correlations beyond trace formula. Plateau?